Data assimilation on the Kalman filter

Jeremias Garay

Introduction

Stationary Case: Least square estimation

Assume we want to find an estimator $\hat{X}$ of a unknown vector X, with a certain guess available $\hat{X}^-$, associated with a confidence matrix $(P^-)^{-1}$. Assume also that we have partial observation Z, satisfying $Z = HX + \zeta^Z$, associated with a confidence matrix W^{-1}.

Stationary Case: Least square estimation

Assume we want to find an estimator $\hat{X}$ of a unknown vector X, with a certain guess available $\hat{X}^-$, associated with a confidence matrix $(P^-)^{-1}$. Assume also that we have partial observation Z, satisfying $Z = HX + \zeta^Z$, associated with a confidence matrix W^{-1}.

A quantity taking care of $\hat{X}^-$ and Z can be obtained minimizing the cuadratic cost functional:

Stationary Case: Least square estimation

Assume we want to find an estimator $\hat{X}$ of a unknown vector X, with a certain guess available $\hat{X}^-$, associated with a confidence matrix $(P^-)^{-1}$. Assume also that we have partial observation Z, satisfying $Z = HX + \zeta^Z$, associated with a confidence matrix W^{-1}.

A quantity taking care of $\hat{X}^-$ and Z can be obtained minimizing the cuadratic cost functional:

$$J(\hat{X}) = \frac{1}{2}(\hat{X} - \hat{X}^-)(P^-)^{-1}(\hat{X} - \hat{X}^-) + \frac{1}{2}(Z - H\hat{X})W^{-1}(Z - H\hat{X}) \qquad (1)$$

Stationary Case: Least square estimation

Find the optimal state imposing: $\frac{dJ}{d\hat{X}}(\hat{X}^{+}) = 0$:

Stationary Case: Least square estimation

Find the optimal state imposing: $\frac{dJ}{d\hat{X}}(\hat{X}^{+}) = 0$:

$$-H^T W^{-1} Z + H^T W^{-1} H \hat{X} - (P^{-})^{-1} \hat{X}^{-} + (P^{-})^{-1} \hat{X} \equiv 0$$

Stationary Case: Least square estimation

Find the optimal state imposing: $\frac{dJ}{d\hat{X}}(\hat{X}^+) = 0$:

$$-H^T W^{-1} Z + H^T W^{-1} H\hat{X} - (P^-)^{-1}\hat{X}^- + (P^-)^{-1}\hat{X} \equiv 0$$

or reordering terms:

Stationary Case: Least square estimation

Find the optimal state imposing: $\frac{dJ}{d\hat{X}}(\hat{X}^+) = 0$:

$$-H^T W^{-1} Z + H^T W^{-1} H\hat{X} - (P^-)^{-1}\hat{X}^- + (P^-)^{-1}\hat{X} \equiv 0$$

or reordering terms:

$$\hat{X}^+ = \hat{X}^- + K(Z - H\hat{X}^-)$$

With $K = P^+ H^T W^{-1}$ the Kalman matrix and $P^+ = ((P^-)^{-1} + H^T W^{-1} H)^{-1}$.

Time dependent problems

The method could be easily expanded into time-dependent systems ($\dot{X} = AX + F$):

Time dependent problems

The method could be easily expanded into time-dependent systems ($\dot{X} = AX + F$):

1. Assume that $\hat{X}^+_{n-1}$ is known with a covariance P^+_{n-1}

Prediction

$$\hat{X}^-_n = A_n \hat{X}^+_{n-1} + F_n$$

by linearity of A_n, the covariance of $\hat{X}^-_n$ is equal to $A_n P^+_{n-1} A^T_n$

Time dependent problems

The method could be easily expanded into time-dependent systems ($\dot{X} = AX + F$):

1. Assume that $\hat{X}^+_{n-1}$ is known with a covariance P^+_{n-1}

Prediction

$$\hat{X}^-_n = A_n \hat{X}^+_{n-1} + F_n$$

by linearity of A_n, the covariance of $\hat{X}^-_n$ is equal to $A_n P^+_{n-1} A^T_n$

2. Afterwards

Correction

$$\hat{X}^+_n = \hat{X}^-_n + K_n(Z_n - H_n \hat{X}^-_n)$$

Non-linear problems

1. *Extended Kalman Filter (EKF)*

Non-linear problems

1. *Extended Kalman Filter (EKF)*
   a. Taylor's expansion on the non-linear operator (tangent operators)

Non-linear problems

1. *Extended Kalman Filter (EKF)*
    a. Taylor's expansion on the non-linear operator (tangent operators)
    b. High cost if the Jacobian can be found numerically

Non-linear problems

1. *Extended Kalman Filter (EKF)*
    a. Taylor's expansion on the non-linear operator (tangent operators)
    b. High cost if the Jacobian can be found numerically
    c. Not optimal when the system is highly non-linear

Non-linear problems

1. *Extended Kalman Filter (EKF)*
   a. Taylor's expansion on the non-linear operator (tangent operators)
   b. High cost if the Jacobian can be found numerically
   c. Not optimal when the system is highly non-linear
2. *Unscented Kalman Filter (UKF)*

Non-linear problems

1. *Extended Kalman Filter (EKF)*
   a. Taylor's expansion on the non-linear operator (tangent operators)
   b. High cost if the Jacobian can be found numerically
   c. Not optimal when the system is highly non-linear
2. *Unscented Kalman Filter (UKF)*
   a. Approximate propagation of vectors by propagating suitable particles

Non-linear problems

1. *Extended Kalman Filter (EKF)*
   a. Taylor's expansion on the non-linear operator (tangent operators)
   b. High cost if the Jacobian can be found numerically
   c. Not optimal when the system is highly non-linear
2. *Unscented Kalman Filter (UKF)*
   a. Approximate propagation of vectors by propagating suitable particles
   b. Could be shown that by computing mean and covariance of the particles, a better approx could be reached.

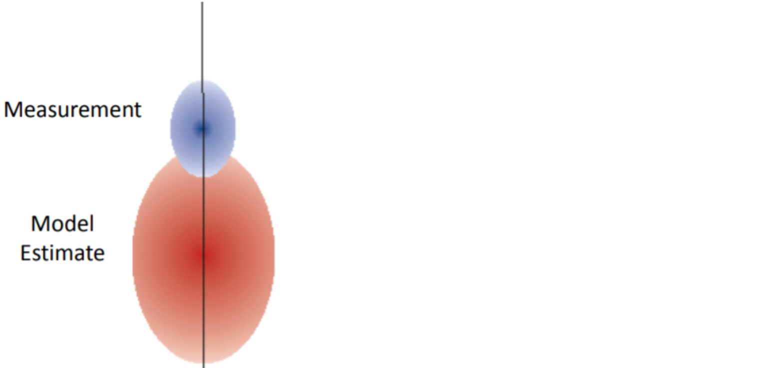
Non-linear problems

1. *Extended Kalman Filter (EKF)*
   a. Taylor's expansion on the non-linear operator (tangent operators)
   b. High cost if the Jacobian can be found numerically
   c. Not optimal when the system is highly non-linear
2. *Unscented Kalman Filter (UKF)*
   a. Approximate propagation of vectors by propagating suitable particles
   b. Could be shown that by computing mean and covariance of the particles, a better approx could be reached.
3. *Reduced Order Unscented Kalman Filter (ROUKF)*

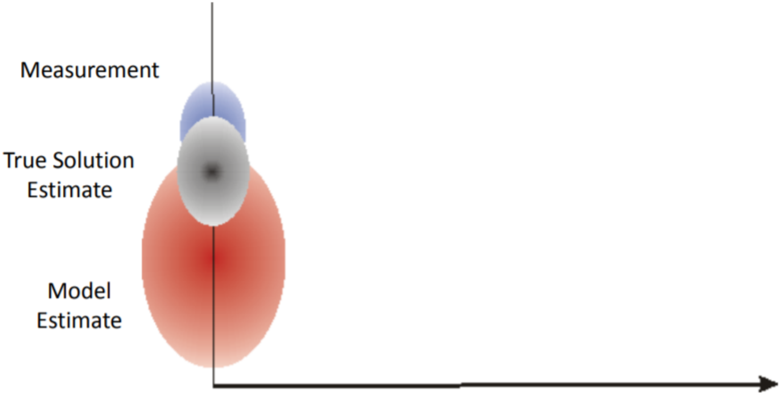
Non-linear problems

1. *Extended Kalman Filter (EKF)*
   a. Taylor's expansion on the non-linear operator (tangent operators)
   b. High cost if the Jacobian can be found numerically
   c. Not optimal when the system is highly non-linear
2. *Unscented Kalman Filter (UKF)*
   a. Approximate propagation of vectors by propagating suitable particles
   b. Could be shown that by computing mean and covariance of the particles, a better approx could be reached.
3. *Reduced Order Unscented Kalman Filter (ROUKF)*
   a. LU factorization could be performed on the covariance matrix P_n^-

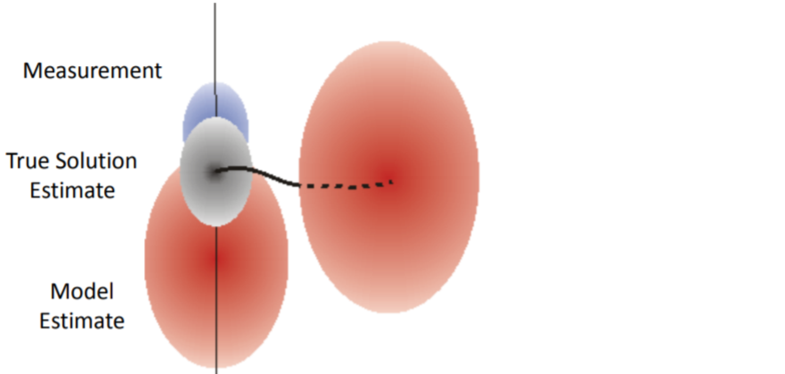
Graphical Picture: Initial State



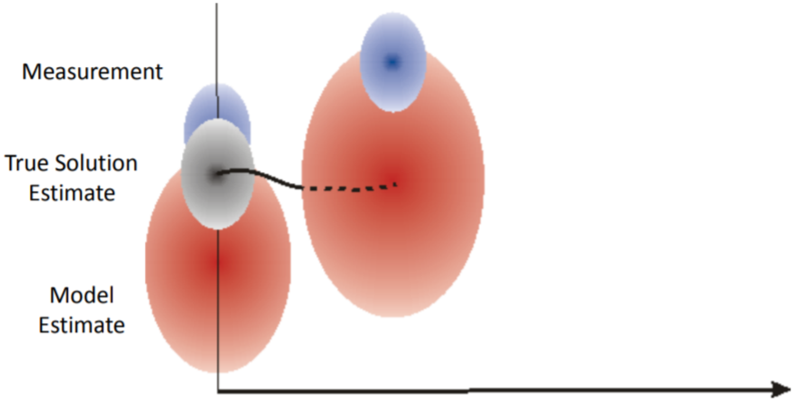
Graphical Picture: Initial State



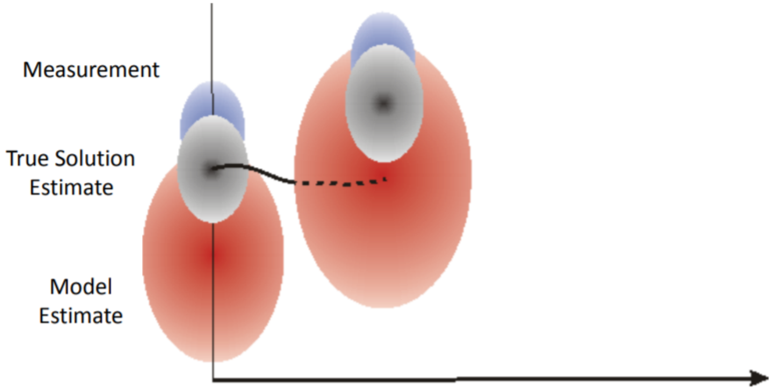
Graphical Picture: Prediction



Graphical Picture: Updating Measurements



Graphical Picture: Correction



Advancing two things: Mean and covariance

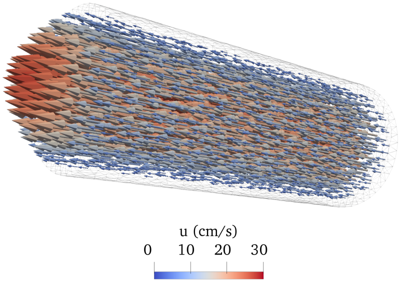
Application: Parameter recovery

Application: Parameter recovery

Consider a Poiseuille flow in a cylinder coming from a simulation. Assume we have:

Application: Parameter recovery

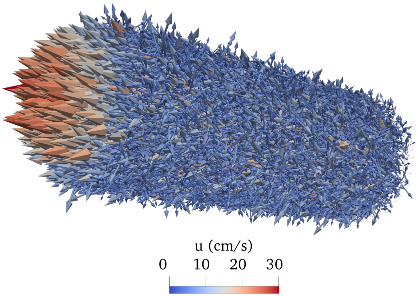
Consider a Poiseuille flow in a cylinder coming from a simulation. Assume we have:



Application: Parameter recovery

Consider a Poiseuille flow in a cylinder coming from a simulation. Assume we have:

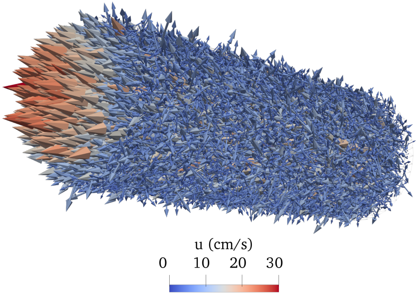
1. The velocity measurements with the addition of some noise



Application: Parameter recovery

Consider a Poiseuille flow in a cylinder coming from a simulation. Assume we have:

1. The velocity measurements with the addition of some noise
2. The measurement's mesh

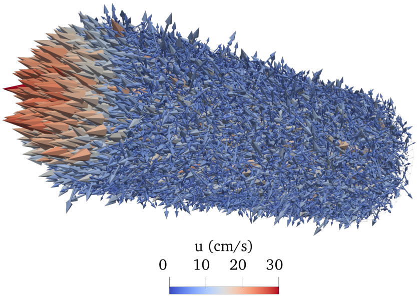


Application: Parameter recovery

Consider a Poiseuille flow in a cylinder coming from a simulation. Assume we have:

1. The velocity measurements with the addition of some noise
2. The measurement's mesh

We want to estimate the amplitude of the inlet flow



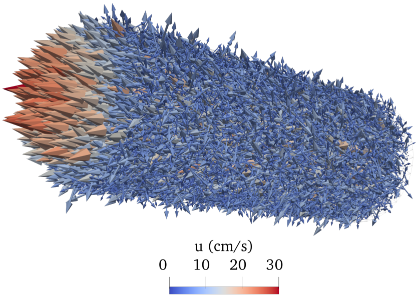
Application: Parameter recovery

Consider a Poiseuille flow in a cylinder coming from a simulation. Assume we have:

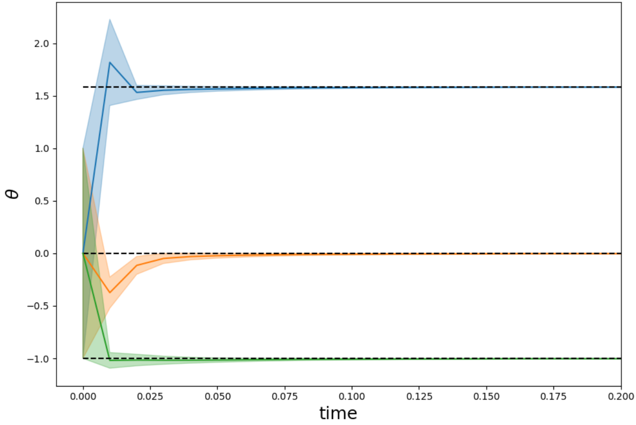
1. The velocity measurements with the addition of some noise
2. The measurement's mesh

We want to estimate the amplitude of the inlet flow

$$u_{inlet} = \color{red}{U}\ (R^2 - r^2)\ sin(\pi t/T)$$



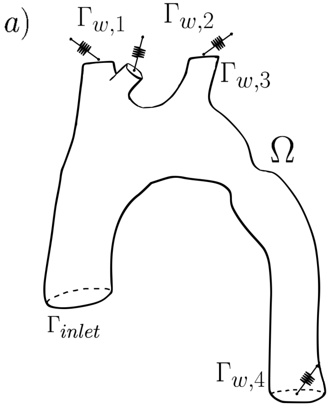
Application: Parameter recovery



Reparametrized value: $\theta_0 \cdot 2^{\theta}$

Application: More complex scenario

Aortic velocity data with reduced order boundary condition:

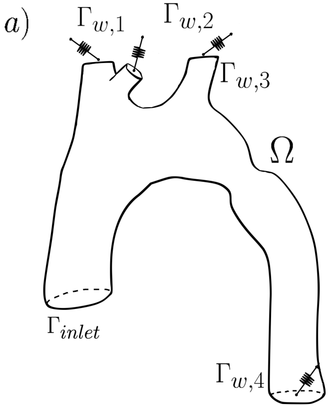


Application: More complex scenario

Aortic velocity data with reduced order boundary condition:

1. Navier-Stokes simulation with a *plug-flow* at the intlet:

$$u_{inlet} = \begin{cases} Usin(\pi t/T) & \text{if } t < T^* \\ \alpha Usin(\pi t/T')e^{-\gamma t} & \text{if } t \geq T^* \end{cases}$$



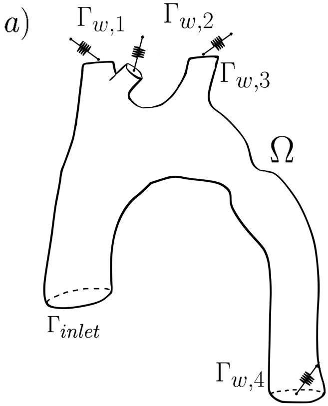
Application: More complex scenario

Aortic velocity data with reduced order boundary condition:

1. Navier-Stokes simulation with a *plug-flow* at the intlet:

$$u_{inlet} = \begin{cases} Usin(\pi t/T) & \text{if } t < T^* \\ \alpha Usin(\pi t/T')e^{-\gamma t} & \text{if } t \geq T^* \end{cases}$$

2. A 1-element Windkessel boundary condition is defined in every inlet.



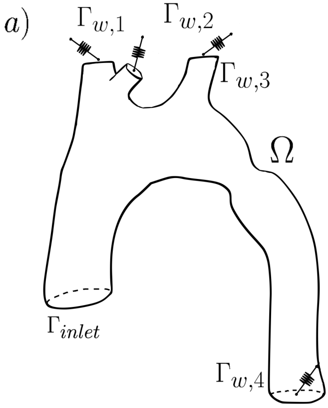
Application: More complex scenario

Aortic velocity data with reduced order boundary condition:

1. Navier-Stokes simulation with a *plug-flow* at the intlet:

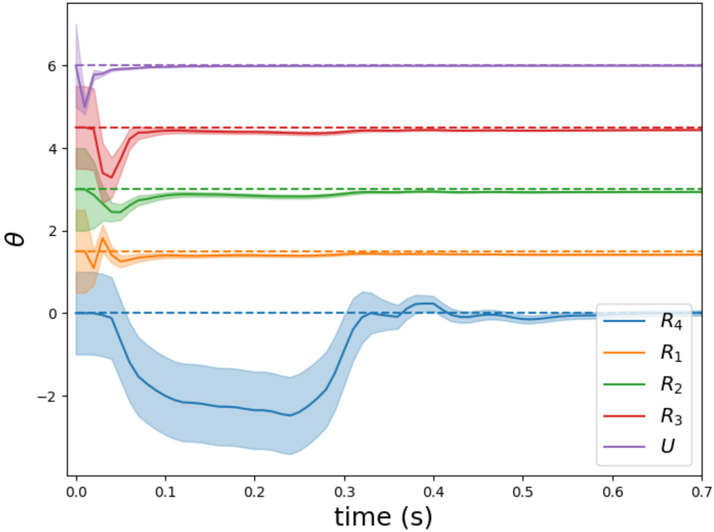
$$u_{inlet} = \begin{cases} Usin(\pi t/T) & \text{if } t < T^* \\ \alpha Usin(\pi t/T')e^{-\gamma t} & \text{if } t \geq T^* \end{cases}$$

2. A 1-element Windkessel boundary condition is defined in every inlet.



We want to recover the proximal resistances R_i, $i = 1, 2, 3, 4$ and the amplitude U from noisy velocity measurements.

Application: Parameter recovery θ_0 2^θ

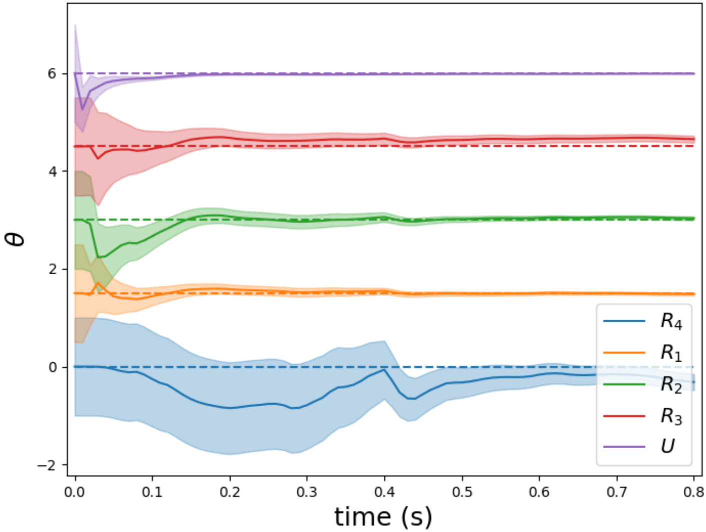


Application: Parameter recovery

	true	*recovered*
$R_1\ (dyn \cdot s \cdot cm^{-5})$	250	242.14
$R_2\ (dyn \cdot s \cdot cm^{-5})$	250	249.16
$R_3\ (dyn \cdot s \cdot cm^{-5})$	250	246.03
$R_4\ (dyn \cdot s \cdot cm^{-5})$	10	9.87
$U\ (cm/s)$	30	29.94

Application: Parameter recovery (only using 1 vel. component)

Application: Parameter recovery (only using 1 vel. component)



Application: Parameter recovery

	true	*recovered*	*recovered with reduced vel*
R_1 $(dyn \cdot s \cdot cm^{-5})$	250	242.14	247.31
R_2 $(dyn \cdot s \cdot cm^{-5})$	250	249.16	255.56
R_3 $(dyn \cdot s \cdot cm^{-5})$	250	246.03	277.37
R_4 $(dyn \cdot s \cdot cm^{-5})$	10	9.87	8.03
U (cm/s)	30	29.94	29.80

Summary

Summary

- Kalman's filter uses a series of measurements and produce an estimate in two steps: Prediction and Correction

Summary

- Kalman's filter uses a series of measurements and produce an estimate in two steps: Prediction and Correction
- The Reduced Order Kalman Filter (ROUKF) its a simplification for non-linear problems which generally run faster than others methods. (no derivatives are need it)

Summary

- Kalman's filter uses a series of measurements and produce an estimate in two steps: Prediction and Correction
- The Reduced Order Kalman Filter (ROUKF) its a simplification for non-linear problems which generally run faster than others methods. (no derivatives are need it)
- Parameter recovery its a straightforward application.